Project systems theory

Resit exam 2018–2019, Friday 12 April 2019, 14:00 – 17:00

Problem 1 (6 + 8 + 4 = 18 points)

Consider the model of population dynamics given by

$$\begin{aligned} \dot{x}_1(t) &= \big(\beta_1 - F(x(t))\big)x_1(t), \\ \dot{x}_2(t) &= \big(\beta_2 - F(x(t))\big)x_2(t), \\ \dot{x}_3(t) &= \big(\beta_3 - F(x(t))\big)x_3(t), \end{aligned} \tag{1}$$

where $x_i(t) \in \mathbb{R}$, $i = 1, 2, 3$ denote the populations of three species and $x = [\, x_1 \ x_2 \ x_3 \,]^{\mathrm{T}}$. Here,

$$F(x) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3, \tag{2}$$

for real parameters $\alpha_i > 0$, $i = 1, 2, 3$, denotes the total burden on the environment and $\beta_i > 0$, $i = 1, 2, 3$, denote the natural growth rates for each species. They are assumed to satisfy

$$\beta_1 > \beta_2 > \beta_3 > 0. \tag{3}$$

(a) Apart from the trivial equilibrium $\bar{x} = 0$, show that any other equilibrium point is necessarily of the form $\bar{x}_i > 0$ for some $i \in \{1, 2, 3\}$ and $\bar{x}_j = 0$ for $j \neq i$.

(b) Consider the equilibrium

$$\bar{x} = \begin{bmatrix} \frac{\beta_1}{\alpha_1} & 0 & 0 \end{bmatrix}^{\mathrm{T}} \tag{4}$$

and linearize the system around this equilibrium point.

(c) Is the linearized system (asymptotically) stable?

Problem 2 (4 + 12 + 6 = 22 points)

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{5}$$

with state $x(t) \in \mathbb{R}^3$, input $u(t) \in \mathbb{R}$, and where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{6}$$

(a) Is the system controllable?

(b) Find a nonsingular matrix T and real numbers α_1, α_2, α_3 such that

$$T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}, \qquad T^{-1}B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{7}$$

(c) Use the matrix T from problem (b) to obtain a state feedback of the form $u = Fx$ such that the closed-loop system matrix $A + BF$ has eigenvalues at -2, -2, and -3.

Problem 3 (15 points)

Consider the linear system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2b & -b & -a \end{bmatrix} x(t), \tag{8}$$

where $a, b \in \mathbb{R}$. Determine the values of a and b for which the system is (asymptotically) stable.

Problem 4 (4 + 4 + 4 + 8 = 20 points)

Consider the linear system

$$\dot{x}(t) = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -4 & 7 \\ 1 & -4 & 6 \end{bmatrix} x(t), \qquad y(t) = \begin{bmatrix} 1 & -2 & 2 \end{bmatrix} x(t). \tag{9}$$

(a) Is the system (9) observable?

(b) Give a basis for the unobservable subspace of the system (9).

In the remainder of this problem, consider the linear system

$$\dot{x}(t) = \begin{bmatrix} a-3 & 8-2a \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 2a \\ a \end{bmatrix} u(t), \tag{10}$$

with a a real parameter.

(c) Determine all values of a for which the system (10) is controllable.

(d) Determine all values of a for which the system (10) is stabilizable.

Problem 5 (15 points)

Consider the linear system

$$\dot{x}(t) = Ax(t), \qquad y(t) = Cx(t), \tag{11}$$

with $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, and A and C real matrices. Recall that a matrix $X \in \mathbb{R}^{n\times n}$ is called positive definite if $v^* X v > 0$ for all complex $v \neq 0$ and with v^* the Hermitian transpose of v.

Assume that the matrix pair (A, C) is observable and that there exists a positive definite symmetric matrix X that solves the matrix equation

$$A^{\mathrm{T}} X + XA + C^{\mathrm{T}} C = 0. \tag{12}$$

Prove that this implies that $\sigma(A) \subset \mathbb{C}_-$, i.e., the system is (asymptotically) stable.

Hint. Use the eigenpair (λ, v) satisfying $Av = \lambda v$ and note that this implies $v^* A^{\mathrm{T}} = \bar{\lambda} v^*$ with $\bar{\lambda}$ the complex conjugate of λ.

(10 points free)